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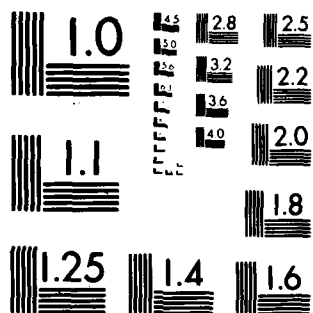
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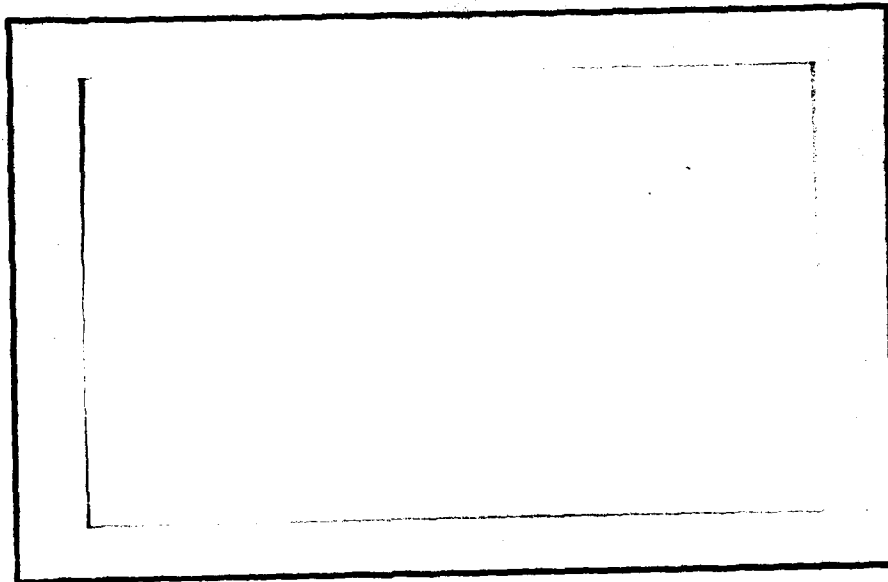


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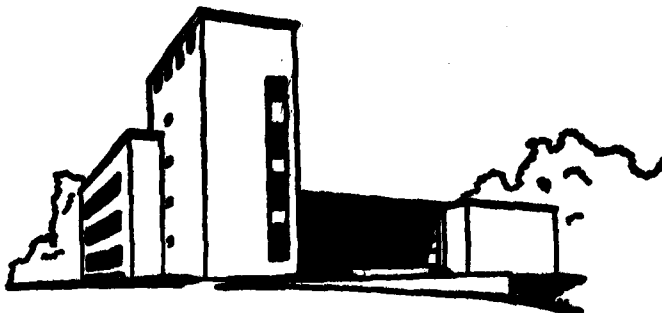
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ON THE CONVERGENCE OF A BLOCK SUCCESSIVE
OVERRELAXATION METHOD FOR A CLASS OF
LINEAR COMPLEMENTARITY PROBLEMS.

by

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ON THE CONVERGENCE OF A BLOCK SUCCESSIVE
OVERRELAXATION METHOD FOR A CLASS OF
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by

R.W. Cottle and J.S. Pang

Abstract. This paper develops a reduced block successive over-relaxation method for solving a class of (large-scale) linear complementarity problems. The main new feature of the method is that it contains certain reduction operations at each iteration. Such reductions are needed in order to ensure the boundedness (and therefore the existence of accumulation points) of the sequence of iterates produced by the algorithm. Convergence of the method is established by using a theorem due to Zangwill.

Key Words: Convergence, block successive overrelaxation algorithms, linear complementarity problem, quadratic programming, compactness, level sets.

Abbreviated title: A block SOR method

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ON THE CONVERGENCE OF A BLOCK SUCCESSIVE
OVERRELAXATION METHOD FOR A CLASS OF
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R.W. Cottle and J.S. Pang

1. Introduction. The present research is motivated by an investigation (still in progress) of methods for solving a certain class of "capacitated quadratic transportation problems". One of these calls for the application of the block successive overrelaxation (BSOR) method [4] to the dual of the given problem. However, a technical problem is engendered by the unboundedness of the level sets of the dual objective function and the consequent breakdown of the convergence proof used in [4]. At issue is the existence of an accumulation point of the sequence of iterates produced by the algorithm. Fortunately, the structure of the problem permits a modification of the algorithm that leads to a remedy for this complication. Applying a simple transformation to the iterates forces the new points to lie in a compact set. Convergence of the algorithm can then be established by invoking a theorem of Zangwill [8].

Our purpose in this paper is to establish the convergence of the modified BSOR for a class of problems somewhat larger than that under consideration in the aforementioned study. To be precise, we concentrate our attention on a (large-scale) linear complementarity problem of the form: Find $y, v \in \mathbb{R}^N$ such that

$$v = f + FAc + FAF'y \geq 0, \quad y \geq 0, \quad v'y = 0. \quad (1)$$

The following blanket assumptions will be maintained throughout this paper:

(A1) The matrix $A \in R^{p \times p}$ is symmetric and positive semi-definite;

(A2) there exists a vector x such that

$$FAx \leq f; \quad (2)$$

(A3) there exists an index set α such that for any y satisfying

$$AF'y = 0$$

$$f'y = 0 \quad (3)$$

$$0 \neq y \geq 0$$

it follows that $y_j > 0$ if and only if $j \in \alpha$.

Remarks. (A3) holds vacuously if (3) has no solution. In fact, the nonexistence of a solution to (3) is equivalent to the so-called Slater condition, i.e. the consistency of the linear inequality system $FAx < f$.

If (A3) holds nonvacuously, the index set α must be nonempty, though its cardinality could be as low as 1. In the capacitated quadratic transportation problem mentioned earlier, (3) does have a solution. There, the introduction of a seemingly mild additional hypothesis on the capacities ensures the validity of (A3), and the index set α is easily identified. It corresponds to the supply and demand constraints of the problem. See Section 6 for further discussion of this application.

Under assumption (A1), the linear complementarity problem (1) is the set of Karush-Kuhn-Tucker conditions for the convex quadratic program

$$\text{minimize } \psi(y) = (f + FAc)'y + \frac{1}{2}y'FAF'y \quad (4)$$

$$\text{subject to } y \geq 0.$$

By a proof like that of Proposition 2.1 below, (A1) and (A2) imply the existence of a solution to (1) or, equivalently, (4).

If the matrix A is in fact positive definite, then (4) is essentially the dual of the strictly convex quadratic program

$$\begin{aligned} \text{minimize} \quad & \phi(x) = c'x + \frac{1}{2}x'A^{-1}x \\ \text{subject to} \quad & Fx \leq f. \end{aligned} \tag{5}$$

Note that (A2) implies the feasibility of (5) and thus the existence of an optimal solution.

In proving the convergence of iterative procedures for nonlinear programming, it is customary to require that the iterates lie in a compact set. The set in question is often a level set of the function being minimized. In the context of the quadratic program (4), the minimand is ψ . As we shall show later, the level sets of ψ are not bounded if the system (3) is consistent.

In the present paper, we shall show how the BSOR method described in [4] can be modified in such a way that the possible unboundedness of level sets will not affect the convergence of the method for solving (4) - or, equivalently, (1). Our analysis provides a unified treatment for both bounded and unbounded level sets. In particular, the analysis includes, as a special case, the recent study of Mangasarian [5] who treats the quadratic program (5) under a Slater condition.

2. Preliminary discussion. Throughout the paper we denote the linear complementarity problem (LCP)

$$w = q + Mz \geq 0, \quad z \geq 0, \quad w'z = 0$$

by the pair (q, M) . For a given M , let $K(M)$ be the set of all vectors q for which (q, M) has a solution. We recall that if M is positive semi-definite, then $q \in K(M)$ if and only if the inequalities

$$q + Mz \geq 0, \quad z \geq 0$$

are consistent. (See [1].)

In the next three results, we present some properties of the LCP (1).

Proposition 2.1. For all vectors a , the linear complementarity problem

$$(f + FAa, FAF') \tag{6}$$

has a solution.

Proof. As FAF' is (symmetric and) positive semi-definite, only consistency need be verified. If the LCP $(f + FAa, FAF')$ is inconsistent, there must exist a vector u such that

$$u'(f + FAa) < 0, \quad u'FAF' \leq 0, \quad u \geq 0.$$

In the presence of (A1), the latter implies

$$u'f < 0, \quad u'FA = 0, \quad u \geq 0. \tag{7}$$

But (A2) implies the existence of a vector x such that (2) holds.

Clearly (2) and (7) cannot both hold simultaneously, so (6) must be consistent. \square

The theorem below characterizes the boundedness (and, consequently, the compactness) of the level sets of certain convex quadratic functions ψ of interest in the present study. Part of its proof relies on a much more general result of Rockafellar.

Theorem 2.1. Suppose (A1) and (A2) are satisfied. For any a , the following statements are equivalent for the quadratic function

$$\psi(y) := (f + FAa)'y + \frac{1}{2}y'FAF'y.$$

(1°) For each $\lambda \in \mathbb{R}$, the set

$$Y(\lambda) := \{y \geq 0: \psi(y) \leq \lambda\}$$

is compact;

(2°) $f + FAa \in \text{int } K(FAF')$;

(3°) there exists no vector y such that

$$AF'y = 0, \quad f'y = 0, \quad 0 \neq y \geq 0; \quad (8)$$

(4°) there exists a vector \tilde{x} such that

$$FA\tilde{x} < f;$$

(5°) the LCP $(f + FAa, FAF')$ has a bounded solution set.

Proof. Define

$$\Psi(y) = \begin{cases} \psi(y) & \text{if } y \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Then (1°) holds if and only if the level sets of Ψ are bounded. By [7, Corollary 14.2.2], this is so if and only if $0 \in \text{int dom } \Psi^*$ where Ψ^* denotes the convex conjugate of Ψ . Now for any y^* we have

$$\begin{aligned}
\psi^*(y^*) &= \sup \{y'y^* - \psi(y) : y \text{ arbitrary}\} \\
&= \sup \{y'y^* - \psi(y) : y \geq 0\} \\
&= -\inf \{y'(q - y^*) + \frac{1}{2}y'My : y \geq 0\} \\
&= \begin{cases} \frac{1}{2}\bar{y}'M\bar{y} & \text{if } \bar{y} \text{ solves } (q-y^*, M) \\ -\infty & \text{if } (q-y^*, M) \text{ has no solution} \end{cases}
\end{aligned}$$

where $q = f + FAa$ and $M = FAF'$. It therefore follows that

$$\text{dom } \psi^* = \{y^* : q - y^* \in K(M)\}$$

From this it is apparent that $0 \in \text{int dom } \psi^*$ if and only if $q \in \text{int } K(M)$. Thus (1°) is equivalent to (2°). It is known from [2] that (2°) is equivalent to the condition that $q \in K(M)$ and the only solution of the system

$$u'q = 0, \quad u'M = 0, \quad u \geq 0 \tag{9}$$

is the zero vector. By Proposition 2.1, $q \in K(M)$ is implied by (A1) and (A2) which are in force here. By the definitions of q and M and the assumed properties of A , (9) becomes

$$u'f = 0, \quad u'FA = 0, \quad u \geq 0.$$

Hence the equivalence of (2°) and (3°) follows. Combined with (A1) and (A2), condition (3°) is equivalent to the fact that

$$u'f \leq 0, \quad u'FA = 0, \quad u \geq 0$$

has only the zero solution. By an alternative theorem, this is equivalent to (4°). The equivalence of (4°) and (5°) is a direct consequence of [2, Theorem 3.1]. \square

Theorem 2.1 has much in common with some characterizations obtained by Mangasarian [6]. Our work along these lines was done independently, however.

Theorem 2.1 implies among other things that if the system in (3°) has a non-zero solution, then the level sets of the function ψ are unbounded. In fact, the following stronger result obviously holds.

Proposition 2.2. Let y^* be any solution of the system

$$AF'y = 0, \quad f'y = 0.$$

Then

$$\psi(y + \theta y^*) = \psi(y)$$

for all y and θ . \square

To describe the BSOR method, we let the rows of the matrix F be partitioned into blocks F_i ($i = 1, \dots, m$). This induces a partitioning of $M = FAF'$ into submatrices $M_{ij} = F_i AF_j'$. Let the vector f be partitioned accordingly. Let J_i denote the set of indices of the rows in F_i (and f_i). Let n_i denote the cardinality of J_i , and finally (referring to (A3)) let

$$\alpha_i = \alpha \cap J_i \quad i = 1, \dots, m.$$

Obviously, the following implication holds:

$$\left. \begin{array}{l} AF_i'y_i = 0 \\ f_i'y_i = 0 \\ 0 \neq y_i \geq 0 \end{array} \right\} \Rightarrow (y_i)_j > 0 \quad \text{if and only if } j \in \alpha_i \quad (10)$$

Once the partitioning above is introduced, then for $i = 1, \dots, m$ Proposition 2.1, Theorem 2.1, and Proposition 2.2 apply to the subproblems $(f_i + F_i Aa, F_i A F_i')$ and quadratic functions

$$\psi_i(y_i) = (f_i + F_i Aa)'y_i + \frac{1}{2}y_i'F_i A F_i'y_i.$$

In particular, as we shall show (Proposition 2.4), assumption (A3) implies that at most one of these subproblems can have an unbounded solution set. Before proving this, we give a geometrical interpretation of the assumption. Let C denote the set of all vectors y satisfying the system (3) and also containing the zero vector. The next result shows that assumption (A3) holds if and only if the set C is a ray emerging from the origin.

Proposition 2.3. Assumption (A3) holds if and only if there exists a non-negative vector y^* such that

$$C = \{y : y = \lambda y^* \text{ for some } \lambda \geq 0\}.$$

Proof. Suppose C is of this form. If the system (3) is inconsistent, there is nothing to prove, so suppose it is consistent. This implies that the vector y^* must be nonzero. Let α be the set of indices which correspond to the nonzero components of y^* (i.e., its support). Obviously, if $y \in C \setminus \{0\}$, then $y_j > 0$ if and only if $j \in \alpha$.

Conversely, suppose that assumption (A3) holds. If (3) is inconsistent it suffices to let y^* be the zero vector. On the other hand, if (3) is consistent, let y^* be any one of its solutions. Let $y \in C \setminus \{0\}$. Consider the vector $y - \lambda y^*$. For suitable $\lambda \geq 0$, the vector $y - \lambda y^*$ will belong to C and have at least one zero component, say the j -th one with $j \in \alpha$. By (A3) this is impossible unless $y - \lambda y^*$ is the zero vector. This proves the proposition. \square

Proposition 2.4. Let assumption (A3) hold. Then for any partitioning of the rows of F , there can exist at most one index i for which the system

$$AF_i'y_i = 0, \quad f_i'y_i = 0, \quad 0 \neq y_i \geq 0 \quad (11)_i$$

is consistent.

Proof. Indeed, if there are indices $i_1 \neq i_2$ for which $(11)_{i_1}$ and $(11)_{i_2}$ are consistent, let $y_{i_1}^*$ and $y_{i_2}^*$ be solutions of these systems, respectively. Obviously, the vectors $y^1 = (y_\ell^1)$ and $y^2 = (y_\ell^2)$ with

$$y_\ell^1 = \begin{cases} 0 & \text{if } \ell \neq i_1 \\ y_{i_1}^* & \text{if } \ell = i_1 \end{cases} \quad y_\ell^2 = \begin{cases} 0 & \text{if } \ell \neq i_2 \\ y_{i_2}^* & \text{if } \ell = i_2 \end{cases}$$

satisfy the system (3). By (A3), we must have $\alpha \subset J_{i_1} \cap J_{i_2} = \emptyset$. This contradiction establishes the proposition. \square

3. Closedness of the component maps. The main tool used in our convergence result for the modified BSR method is Convergence Theorem A of Zhang [8]. To apply the theorem, it is necessary to show that the "algorithmic map" involved is closed. In this section, we establish some preliminary results useful for this purpose.

The total number of rows in the matrix F is $N = \sum_1^m n_i$. For each $i = 1, \dots, m$ let $y_i^* \in R_+^{n_i}$ be either a fixed vector satisfying the system $(11)_i$ if the system is consistent or the zero vector if it is inconsistent. Let $y^* = (y_1^*, \dots, y_m^*)$. Then obviously, y^* is either 0 or it satisfies the system (3).

A vector $y_i \in R_+^{n_i}$ is said to be reduced (with respect to the index set α_i) if at least one component in the subvector $(y_i)_{\alpha_i}$ is equal to zero. Then by $(11)_i$, y_i^* is either zero or not reduced. Let S_i denote the set of all reduced vectors in $R_+^{n_i}$. The i -th reduction map $R_i : R_+^{n_i} \rightarrow S_i$ is defined as follows:

$$R_i(y_i) = y_i - \rho_i y_i^*$$

where

$$\rho_i = \rho_i(y_i) = \begin{cases} \min \{ (y_i)_j / (y_i^*)_j : j \in \alpha_i \} & \text{if } y_i^* \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The reduction map R_i is well defined and continuous. If $y_i^* = 0$, then α_i is just the identity map. Similarly, by dropping the subscript i , we may define the reduced vectors in R_+^N as well as the complete reduction map $R : R_+^N \rightarrow S$, where S is the set of all reduced vectors in R_+^N .

We define the i -th complementarity map $C_i : R_+^N \rightarrow R_+^N \times S_i$ as follows. Given $y = (y_1, \dots, y_m) \in R_+^N$, $C_i(y)$ denotes the set of all points $(y, R_i(\bar{y}_i))$ where \bar{y}_i solves the LCP $(f_i + F_i Aa, F_i A F_i')$ where

$$a = c + \sum_{\ell \neq i} F_\ell' y_\ell.$$

In general, C_i is a point-to-set map. By Proposition 2.1, $C_i(y)$ is nonempty for each i .

Note that if the subproblem has a unique solution, then by Theorem 2.1, $R_i(\bar{y}_i) = \bar{y}_i$ so that the i -th reduction is unnecessary. Roughly speaking, the motivation for including the reduction step in defining the map C_i is to ensure that C_i maps bounded sets into bounded sets.

Let $\omega^* < 2$ be a given positive scalar. Define the i -th relaxation map $P_i : R_+^N \times R_+^{n_i} \rightarrow R_+^N$ as follows. For $(y, \hat{y}_i) \in R_+^N \times R_+^{n_i}$, the set $P_i(y, \hat{y}_i)$ consists of all vectors of the form

$$(y_1, \dots, y_{i-1}, \tilde{y}_i, y_{i+1}, \dots, y_m)$$

where $\tilde{y}_i = y_i + \tilde{\omega}(\hat{y}_i - y_i)$ for some $\tilde{\omega}$ such that

$$\min\{\omega^*, 1\} \leq \tilde{\omega} \leq \omega^* \quad \text{and} \quad \tilde{y}_i \geq 0.$$

The relaxation step in [4] is a particular realization of the relaxation map where $\tilde{\omega}$ is chosen as the largest possible value of ω for which

$$\omega \leq \omega^* \quad \text{and} \quad y_i + \omega(\hat{y}_i - y_i) \geq 0.$$

A point to set map $M : U \rightarrow V$ is bounded if for every subset $T \subset U$, the image $U(M(t) : t \in T)$ is a bounded subset of V .

Let $B_i = P_i \circ C_i$ denote the composition of the i -th complementarity and relaxation maps. In what follows, we show that B_i is a closed and bounded map from R_+^N into itself. We first prove this for C_i .

For each index i and vector $a \in R^D$ let $X_i(a)$ denote the set of all solutions of the LCP $(f_i + FAa, F_iAF_i')$.

Lemma 3.1. $R_i(X_i(a)) = \{y_i \in X_i(a) : \prod_{j \in \Omega_i} (y_i)_j = 0\}$.

Proof. For brevity, let T_i be the set on the right. Since $\rho_i(y_i) = 0$ for each $y_i \in T_i$, it follows that $T_i \subset R_i(X_i(a))$. Conversely, let

$$\hat{y}_i = R_i(\bar{y}_i) \text{ where } \bar{y}_i \in X_i(a). \text{ Then obviously, } \prod_{j \in \Omega_i} (\hat{y}_i)_j = 0.$$

It can easily be shown that \hat{y}_i also solves the LCP $(f_i + FAa, F_iAF_i')$. \square

Proposition 3.1. The i -th complementarity map C_i is both closed and bounded.

Proof. To show that C_i is closed, let

$$y^k \rightarrow y, \quad y^k \in R_+^N$$

$$z^k \rightarrow z = (y, \hat{y}_i), \quad z^k = (y^k, R_i(\bar{y}_i^k)) \in C_i(y^k).$$

As \hat{y}_i is the limit of a sequence of reduced vectors $\bar{y}_i^k = R_i(\bar{y}_i^k)$, it is itself reduced. It therefore suffices to prove that $\hat{y}_i \in X_i(a)$ where

$a = c + \sum_{j \neq i} F_j' y_j$. Lemma 3.1 implies that for each k

$$\hat{y}_i^k \geq 0, \quad \hat{v}_i^k = f_i + F_i A a^k + F_i A F_i' y_i^k \geq 0, \quad (\hat{v}_i^k)' (\hat{y}_i^k) = 0$$

where $a^k = c + \sum_{j \neq i} F_j' y_j^k$. Passing to the limit as $k \rightarrow \infty$, we obtain

$$\hat{y}_i \geq 0, \quad \hat{v}_i = f_i + F_i A a + F_i A F_i' \hat{y}_i \geq 0, \quad (\hat{v}_i)' (\hat{y}_i) = 0.$$

This establishes the closedness of C_i . It is also bounded, for suppose the contrary. Then there exists a bounded subset $T \subset R_+^N$ such that

$U(C_i(t) : t \in T)$ is unbounded. Hence there exists sequences

$\{y^k\} \in T$ and $\{z^k\} = \{(y^k, R_i(\bar{y}_i^k))\}$ with $z^k \in C_i(y^k)$ such that $\|z^k\| \rightarrow \infty$.

Since $\{y^k\}$ is bounded, it has a convergent subsequence $\{y^{k_v}\}$ tending to

some vector $y \in R_+^N$. Let $\hat{y}_i^k = R_i(\bar{y}_i^k)$. Since $\|z^k\| \rightarrow \infty$ and $\{y^k\}$ is

bounded, we must have $\|\hat{y}_i^k\| \rightarrow \infty$. However, the normalized sequence

$\{\hat{y}_i^{k_v} / \|\hat{y}_i^{k_v}\|\}$ has a limit point, \hat{y}_i , and clearly \hat{y}_i is reduced. Without

loss of generality, we may assume $\hat{y}_i^{k_v} / \|\hat{y}_i^{k_v}\| \rightarrow \hat{y}_i$. For each k_v we

have by Lemma 3.1

$$(\hat{y}_i^{k_v})' (f_i + F_i A a^{k_v} + F_i A F_i' \hat{y}_i^{k_v}) = 0$$

where $a^{k_v} = c + \sum_{j \neq i} F_j' y_j^{k_v}$. Dividing the above equation by $\|\hat{y}_i^{k_v}\|^2$ and

passing to the limit we obtain

$$(\hat{y}_i)' F_i A F_i' \hat{y}_i = 0.$$

By (A1) it follows that

$$AF_i' \hat{y}_i = 0. \quad (12)$$

Furthermore, we have

$$0 \geq (\hat{y}_i^{k_v})' (f_i + F_i A a^{k_v})$$

Dividing by $\|\hat{y}_i^{k_v}\|$ and passing to the limit as $k_v \rightarrow \infty$, we obtain (in view of (12))

$$(\hat{y}_i)' f_i \leq 0.$$

By Proposition 2.1, we conclude that \hat{y}_i satisfies

$$0 \neq \hat{y}_i \geq 0, \quad (\hat{y}_i)' f_i = 0, \quad AF_i' \hat{y}_i = 0.$$

Consequently, it follows that $(\hat{y}_i)_{\alpha_i} > 0$. But this contradicts the fact that \hat{y}_i is reduced. \square

Proposition 3.2. The i -th relaxation map P_i is both closed and bounded.

Proof. The boundedness is obvious. To show that P_i is closed,

let $(y^k, \hat{y}_i^k) \rightarrow (y, \hat{y}_i)$ and $z^k \rightarrow z$ where

$$z^k = (y_1^k, \dots, y_{i-1}^k, \hat{y}_i^k, y_{i+1}^k, \dots, y_m^k) \in P_i(y^k, \hat{y}_i^k)$$

and

$$z = (y_1, \dots, y_{i-1}, \hat{y}_i, y_{i+1}, \dots, y_m).$$

It suffices to show that there exists a scalar $\tilde{\omega}$ with $\min\{\omega^*, 1\} \leq \tilde{\omega} \leq \omega^*$ such that $\tilde{y}_i = y_i + \tilde{\omega}(\hat{y}_i - y_i)$. But for each k , there exists a scalar $\tilde{\omega}^k \in [\min\{\omega^*, 1\}, \omega^*]$ such that $\tilde{y}_i^k = y_i^k + \tilde{\omega}^k(\hat{y}_i^k - y_i^k) \geq 0$. Since the $\tilde{\omega}^k$ lie in a compact interval they have a limit point $\tilde{\omega}$. This $\tilde{\omega}$ will do. \square

Lemma 3.2. The composition of two bounded (point-to-set) maps is bounded.

Proof. Indeed, if $M_1 : U \rightarrow V$ and $M_2 : V \rightarrow W$ are two bounded (point-to-set) maps and T is a bounded subset of U , then the set

$$M_2 \circ M_1(T) = \cup \{M_2(s) : s \in M_1(T)\}$$

is obviously bounded. \square

Combining these results, we obtain immediately

Proposition 3.3. The map B_i is both closed and bounded.

Proof. This follows from Propositions 3.1 and 3.2 by applying Lemma 3.2 and [8, Lemma 4.2]. \square

Remark. The boundedness of the complementarity map C_i is crucial in order to apply Lemma 4.2 in [8] to deduce that B_i is closed. For the same reason, the boundedness of B_i is important in proving the closedness of the algorithmic map to be given later. The role played by the reduction maps R_i in these deductions should now be very transparent.

We point out that a vector $z \in B_i(y)$ might not be reduced with respect to α_i . This is because the relaxation map P_i does not necessarily preserve "reducedness".

4. The Reduced BSOR Algorithm. In its simplest form, the modified version of the BSOR method for solving the LCP (1) can be described by its associated algorithmic map

$$A = R \circ B_m \circ \dots \circ B_1 \quad (13)$$

More precisely, given an arbitrary non-negative vector \hat{y}^0 , the algorithm generates a sequence $\{\hat{y}^k\}$ of vectors as follows. If \hat{y}^k solves the problem (1), stop; Otherwise pick a vector $\hat{y}^{k+1} \in A(\hat{y}^k)$ and repeat. For an obvious reason, we call this the Reduced BSOR Algorithm. It is clear that any fixed point of the map A solves the LCP (1).

There are essentially two new features in this Reduced BSOR Algorithm. First, a (possibly unnecessary) reduction is performed after each linear complementarity subproblem is solved. (The precise manner in which these subproblems are solved is optional.) Second, at the end of each iteration, a complete reduction (defined by the map R) is performed. We have seen how reductions of the first kind are useful. Basically, the second kind of reduction is needed for a similar reason; namely, to ensure the boundedness of the sequence $\{\hat{y}^k\}$ generated by the algorithm.

Our principal convergence result for the Reduced BSOR Algorithm is

Theorem 4.1 Applied to the LCP (1) for which (A1), (A2) and (A3) are satisfied, the Reduced BSOR Algorithm either terminates with a

solution or else the sequence of iterates contains an accumulation point which solves the problem.

We first establish three preliminary results. The first one extends Theorem 1 in [4].

Lemma 4.1 Let

$$\phi(x_1, x_2) = \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix}' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $M_{11}, M_{12} = M_{21}', M_{22}, r$, and s are given, M_{11} is symmetric and positive semi-definite, and x_1, x_2 are vector variables.

Let \bar{x}_1 solve the LCP $(r + M_{12}\bar{x}_2, M_{11})$ for some vector \bar{x}_2 . Then for all $x_1 \geq 0$ and all $\omega \in (0, 2)$

$$\phi(x_1 + \omega(\bar{x}_1 - x_1), \bar{x}_2) \leq \phi(x_1, \bar{x}_2)$$

with equality if and only if x_1 also solves the LCP $(r + M_{12}\bar{x}_2, M_{11})$.

Proof. Let $\delta = (\phi(x_1 + \omega(\bar{x}_1 - x_1), \bar{x}_2) - \phi(x_1, \bar{x}_2))/\omega$.

By the proof of Theorem 1 in [4], we have

$$\begin{aligned} \delta &= (\bar{x}_1 - x_1)'(r + M_{12}\bar{x}_2) + (\bar{x}_1 - x_1)'M_{11}x_1 + \frac{\omega}{2}(\bar{x}_1 - x_1)'M_{11}(\bar{x}_1 - x_1) \\ &\leq (\bar{x}_1 - x_1)'(r + M_{12}\bar{x}_2) + (\bar{x}_1 - x_1)'M_{11}x_1 + (\bar{x}_1 - x_1)'M_{11}(\bar{x}_1 - x_1) \\ &= \bar{x}_1'(r + M_{11}\bar{x}_1 + M_{12}\bar{x}_2) - x_1'(r + M_{11}\bar{x}_1 + M_{12}\bar{x}_2) \leq 0. \end{aligned}$$

If $\delta = 0$, then

$$(\bar{x}_1 - x_1)'M_{11}(\bar{x}_1 - x_1) = x_1'(r + M_{11}\bar{x}_1 + M_{12}\bar{x}_2) = 0$$

Since M_{11} is symmetric and positive semi-definite, $M_{11}x_1 = M_{11}\bar{x}_1$ and hence, x_1 solves the linear complementarity problem $(r + M_{12}\bar{x}_2, M_{11})$ as well. \square

Corollary 4.1. For any $y \in R_+^N$ and $z \in A(y)$,

$$\psi(z) \leq \psi(y)$$

with equality if and only if y solves (1).

Proof. This follows easily from the definition of A and repeated use of Proposition 2.2 and Lemmas 3.1 and 4.1. \square

Lemma 4.2. The sequence $\{\hat{y}^k\}$ of iterates generated by the Reduced BSOR algorithm is bounded.

Proof. By Corollary 4.1, we have for each k

$$\psi(\hat{y}^k) \leq \lambda = \psi(\hat{y}^0)$$

The remainder of the proof resembles that of Proposition 4.1 and is omitted. \square

Proof of Theorem 4.1. By repeated use of Proposition 3.3 and [8, Lemma 4.2] one can easily show that the algorithmic map is closed. The desired conclusion now follows from Lemma 4.2, Corollary 4.1 and Convergence Theorem A in [8].

5. An extension. It is rather easy to extend the reduced BSOR algorithm to treat the following generalization of the quadratic program (4):

Find a vector $y \in R^N$ to

$$\text{minimize } \psi(y) = q'y + \frac{1}{2}y'My \text{ subject to } y_i \in Y_i \quad i = 1, \dots, m. \quad (14)$$

Here the vector y is partitioned into subvectors $y_i \in R^{n_i}$ and each Y_i is a nonempty polyhedral set in R^{n_i} :

$$Y_i = \{y_i \in R^{n_i} : B_i y_i \leq b_i \text{ and } E_i y_i = f_i\}$$

where B_i and E_i are arbitrary matrices and b_i and f_i are arbitrary vectors. The matrix M in (14) is symmetric and positive semi-definite and is partitioned into submatrices M_{ij} ($i, j = 1, \dots, m$) where each M_{ij} is n_i by n_j . The vector q is partitioned accordingly.

Without repeating many of the details, we shall in what follows simply present the generalized version of (A3), define the component maps and state the main theorem of convergence for the algorithm. We point out that the program (14) includes as a special case the one treated in [3]. In the latter program, each Y_i is a closed interval of R and the matrix M is symmetric and positive definite.

For $i = 1, \dots, m$, let J_i denote the set of indices in the subvector y_i and let $0^+ Y_i$ denote the recession cone [7] of the set Y_i , i.e.,

$$0^+ Y_i = \{d_i \in R^{n_i} : B_i d_i \leq 0 \text{ and } E_i d_i = 0\}.$$

Define

$$C_i = \{d_i \in R^{n_i} : q_i' d_i = 0, M_{ii} d_i = 0\} \cap 0^+ Y_i$$

and let

$$C = \{d \in R^N : q'd = 0, Md = 0\} \cap \bigcap_{i=1}^m 0^+ Y_i.$$

Let B denote the block diagonal matrix whose diagonal blocks are the B_i .

Finally, let $Y = \prod_{i=1}^m Y_i$ be the feasible set of the program (14). We

state the generalized version of the assumption (A3):

(A4) There exists a nonempty index set α such that for any vector $d \in C \setminus \{0\}$, it follows that $(Bd)_j < 0$ if and only if $j \in \alpha$.

For each $i = 1, \dots, m$, let $\alpha_i = \alpha \cap J_i$. Let d_i^* be a vector in $C_i \setminus \{0\}$ if $C_i \neq \{0\}$ or the zero vector if $C_i = \{0\}$. Let $d^* = (d_1^*, \dots, d_m^*)$.

A vector $y_i \in Y_i$ is said to be reduced (with respect to the index set α_i) if at least one component in the subvector $(b_i - B_i y_i)_{\alpha_i}$ is zero.

Let S_i denote the set of reduced vectors in Y_i . The reduction map $R_i: Y_i \rightarrow S_i$ is defined as follows:

$$R_i(y_i) = y_i - \rho_i d_i^*$$

where

$$\rho_i = \rho_i(y_i) = \begin{cases} \min \{ (B_i y_i - b_i)_j / (B_i d_i^*)_j : j \in \alpha_i \} & \text{if } d_i^* \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, by dropping the subscript i , we may define reduced vectors in Y and the complete reduction map $R: Y \rightarrow S$, where S is the set of reduced vectors in Y .

Extending the i -th complementarity map, we define the i -th sub-program map $S_i: Y \rightarrow Y \times S_i$. Given $y = (y_1, \dots, y_m) \in Y$, $S_i(y)$ denotes the set of all points $(y, R_i(\bar{y}))$ where \bar{y}_i solves the quadratic program

$$\text{minimize } (q_i + \sum_{j \neq i} M_{ij} y_j)' z_i + \frac{1}{2} z_i' M_{ii} z_i \quad \text{subject to } z_i \in Y_i. \quad (15)_i$$

It is important to note that the set Y_i is included as the feasible region of the i -th subprogram.

Finally, the i -th relaxation map $P_i: Y \times Y_i \rightarrow Y$ is defined as follows. Let $\omega^* < 2$ be a given positive scalar. For $(y, \bar{y}_i) \in Y \times Y_i$, the set $P_i(y, \bar{y}_i)$ consists of all vectors of the form

$$(y_1, \dots, y_{i-1}, \tilde{y}_i, y_{i+1}, \dots, y_m)$$

where $\tilde{y}_i = y_i + \tilde{\omega}(\bar{y}_i - y_i)$ for some $\tilde{\omega}$ such that $\min\{\omega^*, 1\} \leq \tilde{\omega} \leq \omega^*$ and $\tilde{y}_i \in Y_i$.

Let $B_i = P_i \circ S_i$ be the i -th component map. The algorithmic map A is defined by (13). The main convergence theorem is the following.

Theorem 5.1. Suppose that the quadratic program (14) has an optimal solution and that assumption (A4) holds. Then, provided that the initial vector \hat{y}^0 is feasible, the same conclusion of Theorem 4.1 holds for the reduced BSOR algorithm applied to the program (14).

Remark. The assumption that the program (14) has an optimal solution is not crucial for the applicability of the algorithm. In fact, without the assumption, the algorithm can still be applied but may terminate at a situation where a certain subprogram $(15)_i$ has an unbounded objective function value. It is easy to show that if this happens, then the original program (14) must have an unbounded objective as well.

6. Concluding Remarks. This paper is intended to provide the theoretical foundations for the Reduced BSOR method which is one of the algorithms being considered in our investigation of computational procedures for solving the capacitated quadratic transportation problem. One of the possible formulations of the latter problem leads to a natural partitioning (of F) with $m = 4$. It also has the property that A is positive definite and diagonal.

Preliminary computational experience with problems of considerable size (e.g. $N \approx 5000$) suggests that the Reduced BSOR method may prove efficient in this application and possibly others as well. We plan to report on our computational results elsewhere.

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